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# Extended Legendre transformation approach to the time-dependent Hamiltonian formalism

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**Abstract.** The Hamiltonian formalism for time-dependent systems is developed using vector fields defined in the extended cotangent bundle $T^*(Q\times\mathbb{R})$. The extended Hamiltonian $\mathbb{H}$ is obtained using the extended Legendre transformation associated to an extended Lagrangian $\mathbb{L}$ previously defined in $T(Q\times\mathbb{R})$ and then the techniques of symplectic mechanics are employed in $T^*(Q\times\mathbb{R})$ to study the properties of the extended Hamiltonian system. Finally this approach is related with the multisymplectic formalism and other extended approaches.

## 1. Introduction

The theory of Hamiltonian dynamical systems on symplectic manifolds $(M, \omega)$ has been shown [1, 2, 3] to be the appropriate geometric setting for the study of autonomous systems in both the Hamiltonian and Lagrangian approaches. In the first case the symplectic manifold is $(T^*Q, \omega_0)$ where $T^*Q$ denotes the cotangent bundle of the configuration space $Q$ and $\omega_0$ the canonical structure of $T^*Q$; in the second case $(M, \omega)$ is $(TQ, \omega_L)$ where $TQ$ is the tangent bundle of $Q$ and $\omega_L$ is constructed from the Lagrangian $L$ by means of the vertical endomorphism [3-5]. Both formalisms are related by the Legendre transformation which establishes, when the Lagrangian $L$ is non-singular, a one-to-one relationship between them.

If the system is time-dependent then the Lagrangian velocity phase space is geometrically represented [1, 2] by $TQ\times\mathbb{R}$ and the momentum phase space by $T^*Q\times\mathbb{R}$. These two manifolds are odd-dimensional and, therefore, they do not admit symplectic structures. Because of this, the symplectic formalism so successfully used for the study of the autonomous Lagrangian and Hamiltonian systems must be left and changed by other geometric formalisms. The approach most often used is the contact (co-symplectic) formalism [1, 2] in manifolds of the form $M\times\mathbb{R}$ with $M$ symplectic. Other structures also used are: vector fields along a map [6], homogeneous and extended formalisms [7-10], multisymplectic formalism [11-15] and other related matters (see [16] and references therein).

In sum, the time-dependent geometric formalism seems to be not so straightforward as the time-independent is, and, although its basic foundations are considered to be clearly stated, some properties that are considered to be well known for the time-independent case (Newtonoid vector fields, Legendre transformation, symmetries and Noether's theorem, Poisson brackets, etc) continue to be studied for the time-dependent case (see, for example, [17-22] for some recent papers). Moreover, one additional

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reason for the study of time-dependent systems is the idea that the geometric formalism of field theories is more closely related to time-dependent classical mechanics than to the autonomous one.

The extended formalism considers [7-9] not $Q$ but the space of events $Q\times\mathbb{R}$ as the configuration space. In this way the new phase spaces, that turn out to be $T(Q\times\mathbb{R})$ and $T^*(Q\times\mathbb{R})$, are even-dimensional and, therefore, suitable for admitting symplectic two-forms. In sum, the extended formalism substitutes a given time-dependent system of $n$ degrees of freedom by an associate time-independent system of $n+1$ degrees of freedom.

The Hamiltonian dynamics can be presented mainly using two ways: (i) directly in the cotangent bundle, (ii) making use of the Legendre transformation. In the first case the Hamiltonian function $H$ is directly given and in the second case $H$ is obtained from a previously defined Lagrangian $L$. In the case of the extended cotangent bundle formalism the two approaches have been considered: thus the extended Hamiltonian $\mathbb{H}$ can be (i) directly constructed in $T^*(Q\times\mathbb{R})$ [8, 9] from a time-dependent Hamiltonian $H$ defined in $T^*Q\times\mathbb{R}$, or (ii) be obtained by means of the homogeneous formalism [7, 10]. The homogeneous Lagrangian $L^h$ of $L$, that is singular in $T(Q\times\mathbb{R})$, is 'almost regular' according to the terminology of Gotay and Nester [23] and of type II [24]. More concretely, the kernel of its associate presymplectic two-form $\omega_L^h$ is two-dimensional [25] with a one-dimensional vertical part $\mathrm{Ker}(\omega_L^h)\cup V[T(Q\times\mathbb{R})]$ generated by the extended Liouville vector field $\Delta'$. In sum, $L^h$ is singular but it can be perfectly studied using the properties of the presymplectic geometry and it has the worth of leading to a Hamiltonian formalism in $T^*(Q\times\mathbb{R})$ by means of a Legendre transformation from $T(Q\times\mathbb{R})$ to $T^*(Q\times\mathbb{R})$.

The purpose of the present paper is present a different approach. We shall study an extended Legendre transformation and the construction of its associate extended Hamiltonian system inside the setting of non-singular systems. The starting point for the first part of the paper is the study of an extended Lagrangian function $\mathbb{L}(L)$ [26] associated to the time-dependent function $L$; nevertheless, we will see that unfortunately, $\mathbb{L}(L)$ is not regular in the whole $T(Q\times\mathbb{R})$ but it presents some singularities.

In section 2 we present the main properties of $\mathbb{L}(L)$ and then in section 3 the Legendre map $D_{\mathbb{L}}$ and the extended Hamiltonian function $\mathbb{H}$ are studied. The remainder of the paper discusses the particular case of the mechanical type systems, the relation of this approach with the multisymplectic formalism and other extended approaches.

## 2. Lagrangian extended formalism

Let $Q'=Q\times\mathbb{R}$ be the new configuration space [7-10]; then the time $t$ will appear as a new coordinate, $q^{n+1}=t$, and the dynamics will be represented by the flow of vector fields defined in $TQ'\approx(TQ\times\mathbb{R})\times\mathbb{R}$. We will denote by $\mu$ the projection $\mu:TQ'\to TQ\times\mathbb{R}$ and by $s$ the parameter of the integral curves; finally the indexes $i,j\dots$ will continue running from 1 to $n$, but $a,b\dots$ will run from 1 to $n+1$.

*Definition 2.1.* Let $Q$ be a manifold and $L:TQ\times\mathbb{R}\to\mathbb{R}$ be a time-dependent regular Lagrangian. Then the function $\mathbb{L}(L):TQ'\to\mathbb{R}$ defined by

$$\mathbb{L}(L)=\mu^*(L)-\mu^*(E_L)(v^{n+1}-1)$$

is called extended Lagrangian associated to $L$.

If there is no danger of confusion $\mathbb{L}(L)$ will be written as $\mathbb{L}$; also in order to simplify the formulae we will omit the $\mu^*$-notation for the $v^{n+1}$-independent functions and write just $L$, $E_L$, etc.

The main properties [26] of $\mathbb{L}$ are:

(1) $\mathbb{L}$ is defined in a tangent bundle and, consequently, it must be considered as a time-independent Lagrangian. Because of this the obtaining of its associate Lagrangian dynamical system must be done by using the two basic objects of the tangent bundle geometry: [3-5] the Liouville vector field $\Delta' \in \mathscr{X}(TQ')$ and the vertical endomorphism $S': \mathscr{X}(TQ') \to \mathscr{X}(TQ')$.

(2) The dynamics is then given by the flow of the Euler-Lagrange vector field $\mathbb{X}_{\mathbb{L}}$ defined as the solution of the Hamiltonian-like equation

$$\mathrm{i}(\mathbb{X}_{\mathbb{L}})\omega_{\mathbb{L}} = \mathrm{d}\mathbb{E}_{\mathbb{L}}$$

where the symplectic form $\omega_{\mathbb{L}}$ and the energy function $\mathbb{E}_{\mathbb{L}}$ are defined by

$$\omega_{\mathbb{L}} = -\mathrm{d}\theta_{\mathbb{L}} \qquad \theta_{\mathbb{L}} = S'^*(\mathrm{d}\mathbb{L}) \qquad \mathbb{E}_{\mathbb{L}} = \Delta'(\mathbb{L}) - \mathbb{L}.$$

Nevertheless the determinant $\mathbb{W}$ of the Hessian matrix $[\partial^2\mathbb{L}/\partial v^a \partial v^b]$ is not constant on $TQ'$ but a velocity-dependent function that takes the form

$$\mathbb{W} = W_{ij}v^i v^j (v^{n+1} - 2) \qquad W_{ij} = \frac{\partial^2 L}{\partial v^i \partial v^j}$$

when $L$ is quadratic. Therefore the two-form $\omega_{\mathbb{L}}$ is symplectic in $TQ'$ up to the points where $\mathbb{W}$ vanishes and, because of this, the vector field $\mathbb{X}_{\mathbb{L}}$ is not defined in the whole $TQ'$ but in the open submanifold $(TQ')_{\mathbb{L}} = TQ' \backslash SQ'$ obtained by removing the singular points $SQ'$. Except for this, $\mathbb{X}_{\mathbb{L}}$ is a second-order differential equation (SODE) field, i.e. $S'(\mathbb{X}_{\mathbb{L}}) = \Delta'$, with a coordinate expression of the form

$$\mathbb{X}_{\mathbb{L}} = v^a \frac{\partial}{\partial q^a} + \mathbb{F}^a(q^b, v^b) \frac{\partial}{\partial v^a}.$$

(3) The equation $v^{n+1} = 1$ defines a $(n+1)$-dimensional submanifold $M = Q' \times \mathbb{R}^n \times \{1\}$ which is an affine subbundle of $TQ'$ trivially diffeomorphic to $TQ \times \mathbb{R}$. $\mathbb{X}_{\mathbb{L}}$ is tangent to $M$ and its restriction $X_M = \mathbb{X}_{\mathbb{L}|M} \in \mathscr{X}(M)$ takes the form

$$X_M = v^k \frac{\partial}{\partial q^k} + W^{kj}F_j \frac{\partial}{\partial v^k} + \frac{\partial}{\partial t}$$

where $F_j(q, v, t)$ are the functions

$$F_j = \frac{\partial L}{\partial q^j} - \frac{\partial^2 L}{\partial q^k \partial v^j} v^k - \frac{\partial^2 L}{\partial t \partial v^j}$$

and $[W^{kj}]$ is the inverse matrix of $[W_{kj}]$. Thus, $X_M$, that is a SODE field in $TQ \times \mathbb{R}$

$$S'(X_M) = 0 \qquad S' = (\mathrm{d}q^i - v^i\,\mathrm{d}t) \otimes \frac{\partial}{\partial v^i}$$

$$\mathrm{i}(X_M)\,\mathrm{d}t = 1$$

turns out to be the suspension [1, 2] of a time-dependent vector field in $TQ$, it agrees with the vector field obtained in the contact formalism and its integral curves satisfy the (time-dependent) Euler-Lagrange equations.

(4) The association $L \to \mathbb{L}(L)$ is natural with respect to gauge transformations.

Summarizing these properties, the Lagrangian introduced in the above definition 2.1 leads to a Lagrangian formalism in which the time $t$ is not treated as a distinguished coordinate and the dynamics is described using the tools of the symplectic tangent bundle geometry. The relation with other best known geometric formalisms is given by the pull-back to the submanifold $M$, that is, when considering the pull-back to $M$ of the 'time-independent' formalism constructed over $\mathbb{L}(L)$ we recover the 'time-dependent' contact formalism constructed directly over $L$.

Unfortunately the carrier space of $\omega_{\mathbb{L}}$ is not the whole $TQ'$ but the open submanifold $(TQ')_{\mathbb{L}} \subset TQ'$. The presence of these singularities represents, no doubt, the most troublesome problem of this extended approach. Later, when we study the mechanical type systems, we will reconsider this question more closely.

## 3. Hamiltonian extended formalism

The Hamiltonian extended formalism uses as momentum phase space the cotangent bundle $T^*Q' \approx (T^*Q \times \mathbb{R}) \times \mathbb{R}^*$; we will denote by $\nu$ the projection $\nu: T^*Q' \to T^*Q \times \mathbb{R}$.

$T^*Q'$ carries, as a consequence of its cotangent structure, a natural or canonical one-form $\Theta_0$ whose exterior derivative gives the symplectic structure

$$\Omega_0 = -\mathrm{d}\Theta_0 \qquad \Theta_0 = p_i\,\mathrm{d}q^i + u\,\mathrm{d}t$$

where $u$ denotes the new momentum, i.e. $u = p_{n+1}$.

In the following $D_{\mathbb{L}}$ will denote the Legendre transformation $D_{\mathbb{L}}: TQ' \to T^*Q'$, $(q^i, t, v^i, v^{n+1}) \to (q^i, t, p_i, u)$, associated with the extended Lagrangian $\mathbb{L}(L)$. Notice that $D_{\mathbb{L}}$ is a map that, although defined in $TQ'$, the domain in which it is a diffeomorphism is given by the aforementioned region obtained by removing the points on which $\omega_{\mathbb{L}}$ is not symplectic. So, from now on, any reference to the diffeomorphic character of this Legendre map is intended to be referred to such open submanifold and to its image in $T^*Q'$.

The image $N = D_{\mathbb{L}}(M)$ of $M$ by the Legendre transformation is a submanfold of $T^*Q'$. We will denote by $\mathrm{i}_N$ the natural injection $\mathrm{i}_N: N \to T^*Q'$.

*Proposition 3.1.* Let $\mathbb{H}$ be the Hamiltonian function $\mathbb{H} = D_{\mathbb{L}*}(\mathbb{E}_{\mathbb{L}})$ obtained from $\mathbb{L}$ and $N$ be defined by $N = D_{\mathbb{L}}(M)$. Then

$$\mathrm{i}_N^*(\mathbb{H}) = 0$$

*Proof.* Since the Legendre transformation $D_{\mathbb{L}}$ is a diffeomorphism, we have

$$D_{\mathbb{L}*} = (D_{\mathbb{L}}^{-1})^*$$

therefore

$$\mathrm{i}_N^*(\mathbb{H}) = \mathrm{i}_N^*(D_{\mathbb{L}*}(\mathbb{E}_{\mathbb{L}})) = \mathrm{i}_N^* \circ (D_{\mathbb{L}}^{-1})^*(\mathbb{E}_{\mathbb{L}}).$$

Let $D_{\mathbb{L}M}$ denote the restriction of the Legendre transformation $D_{\mathbb{L}}$ to the submanifold $M$; then $D_{\mathbb{L}M}$ satisfies $D_{\mathbb{L}} \circ \mathrm{i}_M = \mathrm{i}_N \circ D_{\mathbb{L}M}$. Because of this we obtain

$$\mathrm{i}_N^*(\mathbb{H}) = (D_{\mathbb{L}M}^{-1})^* \circ \mathrm{i}_M^*(\mathbb{E}_{\mathbb{L}}).$$

The energy $\mathbb{E}_L$ takes the form

$$\mathbb{E}_L = \left(\frac{\partial^2 \mathbb{L}}{\partial v^i \partial v^j}\right) v^i v^j (1 - v^{n+1}).$$

Thus

$$i_M^*(\mathbb{E}_L) = 0$$

and hence we obtain

$$i_N^*(\mathbb{H}) = 0. \qquad \square$$

*Proposition 3.2.* The restriction $\Gamma_{\mathbb{H}|N}$ of the Hamiltonian vector field $\Gamma_{\mathbb{H}}$ to the submanifold $N$ defined by $N = D_L(M)$ is a vector field tangent to $N$.

*Proof.* The equation $i(\Gamma_{\mathbb{H}})\Omega_0 = d\mathbb{H}$ determines a Hamiltonian system; then if $e \in \mathbb{R}$ is a regular value of $\mathbb{H}$ (i.e. $d\mathbb{H}(n) \neq 0$ if $n \in \mathbb{H}^{-1}(e)$) the subset $S_e = \mathbb{H}^{-1}(e)$ is a submanifold of $T^*Q'$ of codimension 1 and, by conservation of $\mathbb{H}$, integral curves of $\Gamma_{\mathbb{H}}$ starting in $S_e$ will stay in $S_e$. The submanifold $N$ can be characterized as being the hypersurface $\mathbb{H}^{-1}(0)$; hence it follows that $\Gamma_{\mathbb{H}|N} \in \mathscr{X}(N)$. $\square$

The submanifold $N = D_L(M)$ turns out to be a regular energy surface of the Hamiltonian system and, because of this, the pair $\{N, i_N^*(\Omega_0)\}$ is a contact manifold. The proof is as follows: (i) $i_N^*(\Omega_0)$ is closed since $\Omega_0$ is closed and $d[i_N^*(\Omega_0)] = i_N^*[d(\Omega_0)]$, (ii) $i_N^*(\Omega_0)$ has maximal rank since $\Omega_0$ is non-degenerate and $N$ is of codimension one.

The submanifold $N$ is of dimension $2n+1$ and it can be represented by the equation $v^{n+1}(q, t, p, u) = 1$ which, locally by the implicit function theorem, can be solved for $u = p_{n+1}$ and rewritten as $u = u(q^k, t, p_k)$. This function, $u(q^k, t, p_k)$, represents the coordinate expression of $-(D_{L^*}(E_L))_{|N}$ and it will be denoted by the $-H(q^k, t, p_k)$. Considered as a function defined in $T^*Q'$ it is $\nu$-basic and, therefore [1, 2], it determines a contact form $\omega_H$ in $T^*Q \times \mathbb{R}$; we will denote by $\Omega_H$ the pull-back of $\omega_H$ to $N$.

*Proposition 3.3.* Let $\mathbb{H}$ be the Hamiltonian function $\mathbb{H} = D_{L^*}(\mathbb{E}_L)$ and $N$ defined by $N = D_L(M)$. Then

$$i_N^*(\Omega_0) = \Omega_H.$$

*Proof.* The canonical one-form $\Theta_0$ is

$$\Theta_0 = p_i \, dq^i + u \, dt$$

therefore

$$i_N^*(\Theta_0) = i_N^*(p_i \, dq^i) + i_N^*(u \, dt)$$
$$= p_i \, dq^i - H \, dt.$$

As a consequence of this we obtain

$$i_N^*(\Omega_0) = \Omega_H.$$

Notice that although it seems that the submanifolds $M$ and $N$ represent similar roles in $T(Q \times \mathbb{R})$ and $T^*(Q \times \mathbb{R})$ respectively, they are characterized by very different properties. $M$ is independent of the dynamics, that is, it is the same for all the extended

Lagrangians $\mathbb{L}(L)$. Nevertheless $N$ is dynamical-dependent submanifold, that is, every Hamiltonian function $\mathbb{H}$ determines its own submanfold $N$. Notice also that $N$ can be represented as the graph of a section of the bundle $\nu: T^*Q' \to T^*Q \times \mathbb{R}$. Thus, the time-dependent Hamiltonian $H = H(q^k, t, p_k)$ is obtained as the coordinate representation of the section of $\nu$ (i.e. we identify [6] $C^\infty(T^*Q \times \mathbb{R})$ and $\Gamma(\nu)$) associated with the zero-level surface of $\mathbb{H}$.

Since $\Gamma_{\mathbb{H}}$ is tangent to $N$ its restriction $\Gamma_{\mathbb{H}|N}$ is a characteristic vector field of $i_N^*(\Omega_0)$, i.e. $i(\Gamma_{\mathbb{H}|N})[i_N^*(\Omega_0)] = 0$.

Let us consider a new (extended) Hamiltonian function $\mathbb{H}' \in C^\infty(T^*Q')$ defined [7-10] by

$$\mathbb{H}' = H(q, p, t) + u$$

it determines a new vector field $\Gamma'_{\mathbb{H}} \in \mathscr{X}(T^*Q')$ by the equation

$$i(\Gamma'_{\mathbb{H}})\Omega_0 = d\mathbb{H}'.$$

The constant value surfaces for $\mathbb{H}'$, $S'_r = \mathbb{H}'^{-1}(r)$, are invariant manifolds for $\Gamma'_{\mathbb{H}}$; thus, if $i_r$ represents the natural injection $i_r: S'_r \to T^*Q'$, $\{S'_r, i_r^*(\Omega_0), r \in \mathbb{R}\}$ are contact manifolds. Moreover the restriction of $\Gamma'_{\mathbb{H}}$ to $S'_r$ is a characteristic vector field of $i_r^*(\Omega_0)$.

The crucial point is that $N$, previously characterized as $N = \mathbb{H}^{-1}(0)$, turns out to be also $N = \mathbb{H}'^{-1}(0)$. Therefore both vector fields, $\Gamma_{\mathbb{H}|N}$ and $\Gamma'_{\mathbb{H}|N}$, are characteristic vector fields of the same two-form $\Omega_H = i_N^*(\Omega_0)$

$$i(\Gamma_{\mathbb{H}|N})\Omega_H = 0$$

$$i(\Gamma'_{\mathbb{H}|N})\Omega_H = 0.$$

Consequently $\Gamma_{\mathbb{H}|N}$ and $\Gamma'_{\mathbb{H}|N}$ must be proportional, i.e $\Gamma_{\mathbb{H}|N} = f\Gamma'_{\mathbb{H}|N}$, $f \in C^\infty(N)$. We will prove that they are really the same vector field. The proof is as follows: since $\mathbb{H}' = H(q, t, p) + u$, $\Gamma'_{\mathbb{H}}$ takes the form

$$\Gamma'_{\mathbb{H}} = \frac{\partial H}{\partial p_k}\frac{\partial}{\partial q^k} + \frac{\partial}{\partial t} - \frac{\partial H}{\partial q^k}\frac{\partial}{\partial p_k} - \frac{\partial H}{\partial t}\frac{\partial}{\partial u}$$

and therefore $\langle dt, \Gamma'_{\mathbb{H}} \rangle = 1$ not only in $N$ but in all $T^*Q'$. Concerning $\Gamma_{\mathbb{H}|N}$ we have

$$\langle dt, \Gamma_{H|N} \rangle = i_N^*\left(\frac{\partial \mathbb{H}}{\partial u}\right) = i_N^*(v^{n+1}) = 1.$$

Thus $f = 1$ and $\Gamma_{\mathbb{H}|N} = \Gamma'_{\mathbb{H}|N}$.

On $N$, the dynamics generated by $\mathbb{H}$ is given by the flow of $\Gamma_{\mathbb{H}|N}$ (or $\Gamma'_{\mathbb{H}|N}$) that it is represented by the set of equations

$$\frac{d}{ds} q^k = \frac{\partial H}{\partial p_k} \qquad \frac{d}{ds} p_k = -\frac{\partial H}{\partial q^k}$$

$$\frac{d}{ds} t = 1 \qquad \frac{d}{ds} u = \frac{\partial H}{\partial t}.$$

Notice that $\Gamma_{\mathbb{H}}$ and $\Gamma'_{\mathbb{H}}$ are different vector fields and, therfore, they take different values in $T^*Q'$, $\Gamma_{\mathbb{H}} \neq \Gamma'_{\mathbb{H}}$, and determine different flows. They only coincide on the submanifold $N$ and not in other hypersurfaces corresponding to any other non-zero constant value of the Hamiltonians.

The restriction $\nu_N$ of $\nu$ to $N$ is a contact isomorphism between the contact manifolds $\{N, \Omega_H\}$ and $\{T^*Q\times\mathbb{R}, \omega_H\}$. The projected vector field $\nu_{N*}(\Gamma_{H|N})$, that is the unique characteristic vector field of $\omega_H$ satisfying

$$\langle \mathrm{d}t, \nu_{N*}(\Gamma_{H|N})\rangle = 1$$

turns out to be the suspension [1, 2] $\nu_{N*}(\Gamma_{H|N}) = \tilde{X}_H$ to $T^*Q\times\mathbb{R}$ of the time-dependent vector field

$$X_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}$$

defined in $T^*Q$.

These results have been obtained for Hamiltonians associated to arbitrary regular Lagrangians $L\in C^\infty(TQ\times\mathbb{R})$. Let us study now the case of an standard Lagrangian of mechanical type

$$L(q, v, t) = \tfrac{1}{2}g_{ij}v^iv^j - V$$

where

$$g_{ij} = g_{ij}(q, t) \qquad V = V(q, t).$$

The functions $g_{ij}$ can be considered as the coefficients of a time-dependent Riemannian metric on the $n$-diemsnional manifold $Q$. Notice that the Hessian matrix $W_{ij}$ whose entries are the second derivatives of $L$ with respect to the $n$ velocities $v^i$, $i = 1, \ldots, n$, is $g_{ij}$; consequently Lagrangians of this type are regular.

The extended Lagrangian $\mathbb{L}(L)$ is

$$\mathbb{L}(q^a, v^a) = \tfrac{1}{2}g_{ij}v^iv^j - V - (\tfrac{1}{2}g_{ij}v^iv^j + V)(v^{n+1} - 1)$$

therefore the one-form $\theta_{\mathbb{L}}$ and the energy function $\mathbb{E}_{\mathbb{L}}$ take the form

$$\begin{aligned}\theta_{\mathbb{L}} &= \frac{\partial\mathbb{L}}{\partial v^k}\,\mathrm{d}q^k + \frac{\partial\mathbb{L}}{\partial v^{n+1}}\,\mathrm{d}q^{n+1}\\ &= g_{ij}v^i(2 - v^{n+1})\,\mathrm{d}q^i - (\tfrac{1}{2}g_{ij}v^iv^j + V)\,\mathrm{d}q^{n+1}\\ \mathbb{E}_{\mathbb{L}} &= \frac{\partial\mathbb{L}}{\partial v^k}\,v^k + \frac{\partial\mathbb{L}}{\partial v^{n+1}}\,v^{n+1} - \mathbb{L}\\ &= -g_{ij}v^iv^j(v^{n+1} - 1)\end{aligned}$$

and the momenta are

$$p_k = \frac{\partial\mathbb{L}}{\partial v^k} = g_{ik}v^i(2 - v^{n+1})$$

$$u = \frac{\partial\mathbb{L}}{\partial v^{n+1}} = -(\tfrac{1}{2}g_{ij}v^iv^j + V).$$

The Hessian $\mathbb{W}$ is $\mathbb{W} = g_{ij}v^iv^j\ (v^{n+1} - 2)$, therefore $\mathbb{L}$ has two singularities: the points where the ($n$-dimensional) kinetic energy is null, $g_{ij}v^iv^j = 0$, and $v^{n+1} = 2$. Notice that, although they appear as clearly different, both singularities have the common property of being the points of $TQ'$ where the function $\mathbb{L}$ (that is cubic) reduce to the linear term $\mathbb{L} = -V(q, t)v^{n+1}$. In geometric terms the singularities of $\mathbb{L}$ are $SQ' = S^1Q' \cup S^2Q'$ with

$$S^1Q' = \bigcup S^1_{(q,t)} \qquad S^1_{(q,t)} = \{(v^i, v^{n+1}) \in T_{(q,t)}Q' | v^1 = \ldots = v^n = 0\}$$

$$S^2Q' = \bigcup S^2_{(q,t)} \qquad S^2_{(q,t)} = \{(v^i, v^{n+1}) \in T_{(q,t)}Q' | v^{n+1} = 2\}.$$

Note that these singularities affect only the fibres (i.e. to the velocities) and do not restrict the configuration space $Q'$. It can be proved that $S^1Q'$ is a singularity for the function $\mathbb{F}^{n+1}$ but not for the $\mathbb{F}^k$s and, conversely, $S^2Q'$ is singularity for the $\mathbb{F}^k$s but not for $\mathbb{F}^{n+1}$. Moreover, in the point $M \cap S^1Q' = \{(v^i, v^{n+1}) \mid v^1 = \ldots = v^n = 0, v^{n+1} = 1\}$ the $\mathbb{F}^k$s are well defined and the null value of $\mathbb{F}^{n+1}$ is obtained by continuity of the restriction of $\mathbb{F}^{n+1}$ to $M$.

The Hamiltonian $\mathbb{H} = D_{\mathbb{L}^*}(\mathbb{E}_\mathbb{L})$ turns out to be

$$\mathbb{H} = -2(V+u)\left\{\sqrt{-\frac{1}{2}\frac{g^{ij}p_ip_j}{V+u}} - 1\right\}$$

where $g^{ij}$ is the inverse matrix of $g_{ij}$. For obtaining $\mathbb{H}$ we have made use of

$$g^{ij}p_ip_j = g_{ij}v^iv^j(2-v^{n+1})^2$$

that implies

$$(2-v^{n+1})^2 = -\frac{1}{2}\frac{g^{ij}p_ip_j}{V+u}$$

consequently the image $N$ of $v^{n+1} = 1$ is given by the hypersurface represented by the equation

$$\tfrac{1}{2}g^{ij}p_ip_j + V + u = 0.$$

The Hamiltonian vector field $\Gamma_\mathbb{H}$ reads

$$\Gamma_\mathbb{H} = \frac{\partial\mathbb{H}}{\partial p_k}\frac{\partial}{\partial q^k} + \frac{\partial\mathbb{H}}{\partial u}\frac{\partial}{\partial t} - \frac{\partial\mathbb{H}}{\partial q^k}\frac{\partial}{\partial p_k} - \frac{\partial\mathbb{H}}{\partial t}\frac{\partial}{\partial u}$$

where

$$\frac{\partial\mathbb{H}}{\partial p_k} = \sqrt{-\frac{2W}{p^2}}\,(g^{ki}p_i)$$

$$\frac{\partial\mathbb{H}}{\partial u} = 2 - \sqrt{-\frac{p^2}{2W}}$$

$$\frac{\partial\mathbb{H}}{\partial q^k} = \frac{1}{2}\sqrt{-\frac{2W}{p^2}}\left(\frac{\partial g^{ij}}{\partial q^k}p_ip_j\right) + \left[2 - \sqrt{-\frac{p^2}{2W}}\right]\frac{\partial V}{\partial q^k}$$

$$\frac{\partial\mathbb{H}}{\partial t} = \frac{1}{2}\sqrt{-\frac{2W}{p^2}}\left(\frac{\partial g^{ij}}{\partial t}p_ip_j\right) + \left[2 - \sqrt{-\frac{p^2}{2W}}\right]\frac{\partial V}{\partial t}$$

where we have used the notation $p^2 = g^{ij}p_ip_j$ and $W = V + u$.

The expression of the restriction $\Gamma_{\mathbb{H}|N} \in \mathscr{X}(N)$ is obtained by taking the $\mathrm{i}_N$-pull-back of the above partial derivatives. We obtain

$$\mathrm{i}_N^*\left(\frac{\partial\mathbb{H}}{\partial p_k}\right) = g^{ki}p_i$$

$$\mathrm{i}_N^*\left(\frac{\partial\mathbb{H}}{\partial u}\right) = 1$$

$$\mathrm{i}_N^*\left(\frac{\partial\mathbb{H}}{\partial q^k}\right) = \frac{1}{2}\frac{\partial g^{ij}}{\partial q^k}p_ip_j + \frac{\partial V}{\partial q^k}$$

$$\mathrm{i}_N^*\left(\frac{\partial\mathbb{H}}{\partial t}\right) = \frac{1}{2}\frac{\partial g^{ij}}{\partial t}p_ip_j + \frac{\partial V}{\partial t}$$

and consequently $\Gamma_{\mathbb{H}|N}$ becomes

$$\Gamma_{\mathbb{H}|N}=g^{ki}p_i\frac{\partial}{\partial q^k}+\frac{\partial}{\partial t}-\left(\frac{1}{2}\frac{\partial g^{ij}}{\partial q^k}p_ip_j+\frac{\partial V}{\partial q^k}\right)\frac{\partial}{\partial p_k}-\left(\frac{1}{2}\frac{\partial g^{ij}}{\partial t}p_ip_j+\frac{\partial V}{\partial t}\right)\frac{\partial}{\partial u}$$

that is $\Gamma_{\mathbb{H}|N}=\Gamma'_{\mathbb{H}|N}$ with $\mathbb{H}'=H+u$ defined by

$$\mathbb{H}'=\tfrac{1}{2}g^{ij}(q,t)p_ip_j+V(q,t)+u.$$

Finally, $\Gamma_{\mathbb{H}|N}$ determines in $T^*Q\times\mathbb{R}$ the vector field

$$\nu_{N*}(\Gamma_{\mathbb{H}|N})=\tilde{X}_H=g^{ki}p_i\frac{\partial}{\partial q^k}-\left(\frac{1}{2}\frac{\partial g^{ij}}{\partial q^k}p_ip_j+\frac{\partial V}{\partial q^k}\right)\frac{\partial}{\partial p_k}+\frac{\partial}{\partial t}$$

whose integral curves satisfy

$$\frac{\mathrm{d}}{\mathrm{d}s}q^k=g^{ki}p_i \qquad \frac{\mathrm{d}}{\mathrm{d}s}p_k=-\frac{1}{2}\frac{\partial g^{ij}}{\partial q^k}p_ip_j-\frac{\partial V}{\partial q^k} \qquad \frac{\mathrm{d}}{\mathrm{d}s}t=1$$

that represent the equations of the usual time-dependent dynamics.

Concerning the singularities of $\Gamma_{\mathbb{H}}$ in $T^*Q'$, the following points summarize some of the main characteristics:

(i) Since $\mathbb{L}$ is cubic, the momenta $p_k$ and $u$ are quadratic functions of the velocities $v^a$. Because of this the diffeomorphic character of $D_{\mathbb{L}}$ is local.

(ii) More concretely the points $(v^k, v^{n+1})$ and $(-v^k, 4-v^{n+1})$ have the same image under $D_{\mathbb{L}}$. This means that in every point $(q,t)\in Q'$ the tangent space $T_{(q,t)}Q'$ is divided in two regions $\{v^k, v^{n+1}>2\}$ and $\{v^k, v^{n+1}<2\}$ which give the same image under $D_{\mathbb{L}}$ and have the singularity $S^2_{(q,t)}$ (i.e. $v^{n+1}=2$) as common boundary.

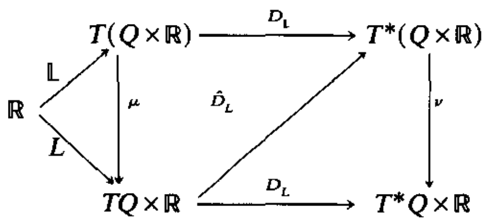
(iii) The definition of $u$ shows that in every point $(q,t)\in Q'$ the momentum $u$ is bounded by the potential, $u\leqslant -V(q^k,t)$, the points $(p_k,u)$ such that $u+V=0$ corresponding to the boundary of the phase space. Every one of the two regions in $T_{(q,t)}Q'$ can be foliated by the family of hyperplanes $v^{n+1}=k$ where $k$ is a constant. The image in $T^*Q'$ by $D_{\mathbb{L}}$ is the family of parabolic-like submanifolds $(\frac{1}{2})\,g^{ij}p_ip_j+(2-k)^2(V+u)=0$. When $k$ takes the value $k=2$ the submanifold collapse the $p_1=\ldots=p_n=0$ and when we take the asymptotic limit $k\to-\infty$ we obtain $u+V=0$. That is, the phase space for $\Gamma_{\mathbb{H}}$ is $Q'\times\{\mathbb{R}^n-\{0\}^n\}\times(-\infty,-V)$ which is the open subset of $T^*Q'$ defined by the inequality $u+V<0$ with the points $p_1=\ldots=p_n=0$ excluded. Note that in this region the square roots appearing in the coordinate expression of $\Gamma_{\mathbb{H}}$ are well defined.

In sum, the singularities of $\Gamma_{\mathbb{H}}$ in $T^*Q'$ are direct consequence of the cubic character of $\mathbb{L}$ (quadratic for $D_{\mathbb{L}}$) and they correspond to those points where the ($n$-dimensional) kinetic energy vanishes, that is, $p_1=\ldots=p_n=0$ and $u+V=0$. Nevertheless, it must be noted that they are singularities for $\Gamma_{\mathbb{H}}$ in $T^*Q'$ but not $\Gamma_{\mathbb{H}|N}$ which is well defined in the whole $N$.

We conclude this section by considering the relation of this approach with the multisymplectic formalism.

An alternative geometric description of the time-dependent systems can be obtained as a very special case of the multisymplectic formalism [11-15] developed for field theories. Starting with a bundle $\pi:E\to M$, $\dim M=m$, one can define for every function $L:J^1\pi\to\mathbb{R}$ a map $\hat{D}_L:J^1\pi\to\Lambda^m_1(E)$ (here $J^1\pi$ is the 1-jet bundle of $\pi$ and $\Lambda^m_1(E)$ denotes the subbundle of $\Lambda^m(E)$ of those of $m$-forms that give zero when two of their arguments are $\pi$-vertical vector fields) and using it one can go to the dual and construct a geometric Hamiltonian formalism for field theories. The point is to consider $\mathbb{R}$ as the basic parameter space (i.e. $M=\mathbb{R}$) and the trivial bundle $\pi:Q\times\mathbb{R}\to\mathbb{R}$ as the fundamental bundle of the approach. Then $J^1\pi \approx TQ\times\mathbb{R}$ and the map $\hat{D}_L$ becomes $\hat{D}_L: TQ\times\mathbb{R}\to T^*Q'$, $(q^i, t, v^i)\to(q^i, t, \partial L/\partial v^i, -E_L)$. So, in this formalism the fundamental map is $\hat{D}_L$ and the usual time-dependent Legendre transformation $D_L$ (that is called 'restricted' Legendre transformation) is obtained as the composite map $\nu\circ\hat{D}_L$.

Let $L$ be a time-dependent regular Lagrangian, then $D_L$ is an imbedding of $TQ\times\mathbb{R}$ into $T^*Q'$ and the relation between the extended Legendre transformation constructed over $\mathbb{L}(L)$ and the multisymplectic formalism constructed over $L$ can be summarized in the following diagram of Legendre maps,




Since $D_{\mathbb{L}}$ is not bundle morphism form $\mu$ to $\nu$, this diagram is not commutative (i.e. $\nu\circ D_{\mathbb{L}}\neq D_L\circ\mu$) but $D_{\mathbb{L}}(M)=\hat{D}_L(TQ\times\mathbb{R})$ and therefore $\nu\circ D_{\mathbb{L}|M}=D_L\circ\mu_{|M}$.

## 4. Final comments

The properties of $\mathbb{L}(L)$ given above in section 2 have been supplemented with an appropriate (extended) Hamiltonian formalism. Moreover, one interesting property is that the (extended) Lagrangian $\mathbb{L}(L)$ leads not only to its own Hamiltonian $\mathbb{H}$ but also (indirectly) to $\mathbb{H}'$. These two extended Hamiltonians, $\mathbb{H}$ and $\mathbb{H}'$, represent two different ways of approaching the extended theory of Hamiltonian systems. Both functions are defined in the cotangent bundle $T^*Q'$ and consequently, both must be considered as 'time-independent' Hamiltonians. Nevertheless they lead to two Hamiltonian formalisms characterized by different properties:

(*a*) $\mathbb{H}'$ (that represents [7–10] the usual Hamiltonian of the extended formalism) can be directly defined in $T^*Q'$ by just pulling back $H$ and adding $u$.

(*b*) $\mathbb{H}$ arises from the extended Lagrangian $\mathbb{L}(L)$ by the use of the extended Legendre Transformation.

(*c*) The dynamics defined by $\mathbb{H}'$ is given by a $\nu$-projectable vector field; thus we recover the usual time-dependent formalism by projecting its Hamiltonian vector field $\Gamma'_{\mathbb{H}}$ on $T^*Q\times\mathbb{R}$.

(*d*) The dynamics defined by $\mathbb{H}$ is not $\nu$-projectable; thus the usual time-dependent formalism is recovered (as was the case for the Lagrangian $\mathbb{L}$) not by projection but for restriction (vector fields) or pull-back (forms).

Finally, notice also that $\mathbb{H}'$ is linear in $u$ and therefore its Legendre transformation $D'_{\mathbb{H}}: T^*Q'\to TQ'$ presents problems. In fact the equation $\mathbb{H}'=0$ can be considered [7, 10] as the Hamiltonian constraint determined in $T^*Q'$ by the extended 'homogeneous Lagrangian' $L^h$ of $L$. The extended Hamiltonian $\mathbb{H}$ has also some unusual characteristics as consequence of its non-polynomical character (i.e. it contains a square root) but its Legendre transformation $D_{\mathbb{H}}: T^*Q'\to TQ'$ can be studied as a local diffeomorphism and the Lagrangian $\mathbb{L}(L)$ can be recovered by

$$\mathbb{L}(L)=D_{\mathbb{H}*}\{\mathrm{i}(\Gamma_{\mathbb{H}})\Theta_0\}-D_{\mathbb{H}*}(\mathbb{H}).$$

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